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# On the nature of fermion-monopole supersymmetry

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## Abstract

It is shown that the generator of the nonstandard fermion-monopole supersymmetry uncovered by De Jonghe, Macfarlane, Peeters and van Holten, and the generator of its standard  $N = 1/2$  supersymmetry have to be supplemented by their product operator to be treated as independent supercharge. As a result, the fermion-monopole system possesses the nonlinear  $N = 3/2$  supersymmetry having the nature of the 3D spin-1/2 free particle's supersymmetry generated by the supercharges represented in a scalar form. Analyzing the supercharges' structure, we trace how under reduction of the fermion-monopole system to the spherical geometry the nonlinear  $N = 3/2$  superalgebra comprising the Hamiltonian and the total angular momentum as even generators is transformed into the standard linear  $N = 1$  superalgebra with the Hamiltonian to be the unique even generator.

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**1.** Some time ago, De Jonghe, Macfarlane, Peeters and van Holten uncovered [1] that in addition to the standard  $N = 1/2$  supersymmetry [2], the fermion-monopole system has the hidden supersymmetry of the nonstandard form. The latter is characterized by the supercharge anticommuting for the nonlinear operator different from the Hamiltonian and equal to the (shifted for the constant) square of the total angular momentum operator. The nonlinear supersymmetry of the similar form was observed also under investigation of the space-time symmetries in terms of the motion of pseudoclassical spinning point particles [3, 4, 5], and was revealed in the 3D  $P, T$ -invariant systems of relativistic fermions [6] and Chern-Simons fields [7]. In this letter we show that the generators of the nonstandard [1] and standard [2] supersymmetries have to be supplemented by their product operator to be treated as independent supercharge. As a result, the fermion-monopole system possesses the nonlinear  $N = 3/2$  supersymmetry having the nature of the 3D spin-1/2 free particle's supersymmetry generated by the supercharges represented in a scalar form. So, the recent observation of the essentially free nature of the dynamics of the scalar charged particle in a monopole field [8] is extended for the fermion-monopole system in the context of supersymmetry. Analyzing the supercharges' structure, we trace how under reduction of the fermion-monopole system to the spherical geometry the nonlinear  $N = 3/2$  superalgebra comprising the Hamiltonian and the total angular momentum as even generators is transformed into the standard linear  $N = 1$  superalgebra with the Hamiltonian to be the unique even generator.

The paper is organized as follows. We start with the analysis of the supersymmetry of the 3D spin-1/2 free particle, and then investigate the fermion-monopole supersymmetry and discuss the reduction of the latter system to the spherical geometry. In conclusion we compare the structure of supersymmetries of the free fermion and fermion-monopole systems with the structure of the  $N = 1$  supersymmetric quantum mechanical systems [9].

**2.** Let us consider a 3D free spin-1/2 nonrelativistic particle given by the classical Lagrangian

$$L = \frac{1}{2}\dot{\mathbf{r}}^2 + \frac{i}{2}\psi\dot{\psi}. \quad (1)$$

The corresponding Hamiltonian is  $H = \frac{1}{2}\mathbf{p}^2$ , and nontrivial Poisson-Dirac brackets are  $\{r_i, p_j\} = \delta_{ij}$  and

$$\{\psi_i, \psi_j\} = -i\delta_{ij}. \quad (2)$$

The set of even integrals of motion is given by the vectors  $\mathbf{p}$ ,  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$  and  $\mathbf{K} = \mathbf{p} \times \mathbf{L}$ , which have a nonzero projection on a unit of Grassmann algebra, and by the nilpotent spin vector  $\mathbf{S} = -\frac{i}{2}\psi \times \psi$  generating rotations of odd Grassmann variables  $\psi_i$ . The vector  $\mathbf{K}$  is the analog of the Laplace-Runge-Lenz vector of the Kepler system, which together with  $\mathbf{p}$  and  $\mathbf{L}$  constitute a non-normalized basis of orthogonal vectors forming a nonlinear algebra with nontrivial part given by the relations

$$\{L_i, V_j\} = \epsilon_{ijk}V_k, \quad V_i = p_i, L_i, K_i, \quad \{K_i, p_j\} = \mathbf{p}^2\delta_{ij} - \mathbf{p}_i\mathbf{p}_j, \quad \{K_i, K_j\} = -\mathbf{p}^2\epsilon_{ijk}L_k. \quad (3)$$

The variables  $\psi_i$  form the odd vector integral of motion, and their algebra (2) is the classical analog of Clifford algebra with three generators. Projecting the odd vector  $\psi$  onto the even vector integrals of motion, we get three odd scalar integrals of motion (supercharges)

$$Q_1 = \mathbf{p}\psi, \quad Q'_2 = \mathbf{L}\psi, \quad Q_3 = \mathbf{K}\psi. \quad (4)$$

The supercharge  $Q_1$  is a “square root from the Hamiltonian”,  $\{Q_1, Q_1\} = -2iH$ . It has zero bracket with the supercharge  $Q_3$ ,  $\{Q_1, Q_3\} = 0$ , but  $Q_1$  and  $Q_3$  have nontrivial brackets with  $Q'_2$ , in particular,  $\{Q_1, Q'_2\} = 2i\mathbf{L}\mathbf{S}$ . One can find the linear combination of the odd scalar integrals  $Q'_2$  and  $i\epsilon_{ijk}\psi_i\psi_j\psi_k$  having zero brackets with other two supercharges,

$$Q_2 = \mathbf{L}\psi - \frac{i}{3}(\psi \times \psi) \cdot \psi. \quad (5)$$

Finally, we get the set of three scalar supercharges,  $\{J_i, Q_a\} = 0$ ,  $a = 1, 2, 3$ , forming together with  $H$  and  $\mathbf{J}$  the nonlinear superalgebra

$$\{Q_1, Q_1\} = -2iH, \quad \{Q_2, Q_2\} = -i\mathbf{J}^2, \quad \{Q_3, Q_3\} = -2iH \cdot \mathbf{J}^2, \quad (6)$$

$$\{Q_a, Q_b\} = 0, \quad a \neq b, \quad (7)$$

where  $\mathbf{J} = \mathbf{L} + \mathbf{S}$  is the total angular momentum vector. The scalar supercharges satisfy also the algebraic relations:

$$Q_1 Q_2 = i\tilde{Q}_3, \quad Q_1 Q_3 = -2iH \cdot \tilde{Q}_2, \quad Q_2 Q_3 = i\mathbf{J}^2 \cdot \tilde{Q}_1. \quad (8)$$

Here  $\tilde{Q}_a$  means  $Q_a$  with odd vector  $\psi$  changed for even  $\mathbf{S}$ . Since  $\mathbf{J}^2 \cdot \tilde{Q}_1 = \mathbf{L}^2 \cdot \tilde{Q}_1$ , relations (8) reflect the analogous relations between the even vector integrals  $\mathbf{p}$ ,  $\mathbf{L}$  and  $\mathbf{K}$ :

$$\mathbf{p} \times \mathbf{L} = \mathbf{K}, \quad \mathbf{p} \times \mathbf{K} = -2H \cdot \mathbf{L}, \quad \mathbf{L} \times \mathbf{K} = \mathbf{L}^2 \cdot \mathbf{p}. \quad (9)$$

Taking into account the equalities  $\mathbf{J}^2 = 2H \cdot \Delta$  and

$$\{\Delta, H\} = \{\Delta, Q_a\} = 0, \quad (10)$$

where  $\Delta = \mathbf{r}_\perp^2 + (\mathbf{L}\mathbf{S}) \cdot H^{-1}$ ,  $\mathbf{r}_\perp = \mathbf{r} - \mathbf{p}(\mathbf{p}\mathbf{r}) \cdot \mathbf{p}^{-2} = \mathbf{K} \cdot \mathbf{p}^{-2}$ , one can transform (at  $\mathbf{p}^2 \neq 0$ ,  $\mathbf{K}^2 \neq 0$ ) the set of scalar supercharges  $Q_a$  into the set

$$\bar{Q}_1 = Q_1, \quad \bar{Q}_2 = Q_2 \cdot \Delta^{-1/2}, \quad \bar{Q}_3 = Q_3 \cdot (2H\Delta)^{-1/2}. \quad (11)$$

This set of (nonlinearly) transformed scalar supercharges gives rise to the  $N = 3/2$  linear superalgebra of the standard form,

$$\{\bar{Q}_a, \bar{Q}_b\} = -2iH\delta_{ab}. \quad (12)$$

The quantum analogs of the odd variables can be realized via the Pauli matrices (we use the system of units  $\hbar = c = 1$ ),  $\hat{\psi}_i = \frac{1}{\sqrt{2}}\sigma_i$ , and the quantum spin vector  $\hat{\mathbf{S}}$  is proportional (“parallel”) to  $\hat{\psi}$ :  $\hat{\mathbf{S}} = \frac{1}{2}\boldsymbol{\sigma} = \frac{1}{\sqrt{2}}\hat{\psi}$ . Note that classically this property is reflected in the relation  $\psi \times \mathbf{S} = 0$ . To construct the quantum analogs of the supercharges  $Q_a$  in the form of Hermitian operators, we choose the natural prescription  $\mathbf{p} \times \mathbf{L} \rightarrow \frac{1}{2}(\hat{\mathbf{p}} \times \hat{\mathbf{L}} + (\hat{\mathbf{p}} \times \hat{\mathbf{L}})^\dagger)$ , and get

$$\hat{Q}_1 = \hat{\mathbf{p}}\hat{\psi}, \quad \hat{Q}_2 = \hat{\mathbf{L}}\hat{\psi} + \frac{1}{\sqrt{2}}, \quad \hat{Q}_3 = (\hat{\mathbf{p}} \times \hat{\mathbf{L}})\hat{\psi} - i\hat{Q}_1. \quad (13)$$

These operators are Hermitian and satisfy the quantum relations (there is no summation in repeated indexes)

$$\hat{Q}_a \hat{Q}_b = \frac{1}{2} \hat{A}_a \delta_{ab} + \frac{i}{\sqrt{2}} \hat{B}_c \epsilon_{abc} \hat{Q}_c, \quad (14)$$

$$\hat{A}_1 = 2\hat{H}, \quad \hat{A}_2 = \hat{\mathbf{J}}^2 + \frac{1}{4}, \quad \hat{A}_3 = \hat{A}_1 \hat{A}_2, \quad (15)$$

$$\hat{B}_1 = \hat{A}_2, \quad \hat{B}_2 = \hat{A}_1, \quad \hat{B}_3 = 1. \quad (16)$$

The symmetric part of relations (14) is the nonlinear  $N = 3/2$  superalgebra being the exact quantum analog of classical relations (6), (7), whereas, with taking into account the above mentioned relation between  $\hat{S}_i$  and  $\hat{\psi}_i$ , we find that the antisymmetric part of (14) corresponds to the classical relations (8).

Due to the relation  $\hat{Q}_3 = -i\sqrt{2}\hat{Q}_1\hat{Q}_2$ , it seems that one could interpret  $\hat{Q}_1$  and  $\hat{Q}_2$  as the “primary” supercharge operators and  $\hat{Q}_3$  as the “secondary” operator. But such an interpretation is not correct. Indeed, on the one hand, the relations (14) can be treated as a quantum generalization of the symmetric in indexes relations for the three Clifford algebra generators  $\sigma_i = \sqrt{2}\hat{\psi}_i$ ,  $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ , and we remember that the operators  $\hat{\psi}_i$  are the quantum analogs of the set of the classical odd integrals of motion  $\psi_i$  forming a 3D vector. On the other hand, the antisymmetric part of (14) is a reflection of the cyclic classical relations (9). It is worth noting also that we can pass from the set of odd integrals (11) (constructed at  $\mathbf{p}^2 \neq 0$ ,  $\mathbf{K}^2 \neq 0$ ) to the integrals

$$\xi_a = \bar{Q}_a \cdot (2H)^{-1}. \quad (17)$$

Then, due to the relations  $\{\xi_a, \xi_b\} = -i\delta_{ab}$ , one can treat the transition from the integrals  $\psi_i$  to the integrals  $\xi_a$  as a simple canonical transformation for the odd sector of the phase space (note, however, that  $\xi_a$  have nontrivial brackets with even variables). Formally, the same unitary in the odd sector transformations can be realized at the quantum level. So, it is natural to treat all the three supercharge operators  $\hat{Q}_a$  on the equal footing.

**3.** Let us consider the case of fermion particle with charge  $e$  in arbitrary time-independent magnetic field  $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$ . At the Hamiltonian level this corresponds to the change of the canonical momentum vector  $\mathbf{p}$  for the vector  $\mathbf{P} = \mathbf{p} - e\mathbf{A}$ ,  $\{P_i, P_j\} = e\epsilon_{ijk}B_k$ . By projecting the odd vector variable  $\psi$  onto  $\mathbf{P}$ , we get the scalar  $Q_1 = \mathbf{P}\psi$ . Identifying the bracket  $\frac{i}{2}\{Q_1, Q_1\}$  as the Hamiltonian,

$$H = \frac{1}{2}\mathbf{P}^2 - e\mathbf{B}\mathbf{S}, \quad (18)$$

$Q_1$  is automatically the odd integral of motion (supercharge). However, now, unlike the case of a free particle, either even vectors  $\mathbf{P}$ ,  $\mathbf{L} = \mathbf{r} \times \mathbf{P}$ ,  $\mathbf{P} \times \mathbf{L}$ ,  $\mathbf{S}$  or odd vector  $\psi$  are not integrals of motion, whereas the odd scalar  $(\psi \times \psi) \cdot \psi$  is conserved. Having in mind the analogy with the free particle case, let us check other odd scalars for their possible conservation. First, it is worth noting that the quantum relation  $\hat{\mathbf{S}} = \frac{1}{\sqrt{2}}\hat{\psi}$  is reflected classically also in the form of the identical evolution,  $\dot{\psi} = e\psi \times \mathbf{B}$ ,  $\dot{\mathbf{S}} = e\mathbf{S} \times \mathbf{B}$ . Like the projection of the odd vector  $\psi$ , the projection of  $\mathbf{S}$  on  $\mathbf{P}$  is conserved, i.e. as in a free case,  $\tilde{Q}_1 = \mathbf{P}\mathbf{S}$  is the integral of motion, but generally  $\frac{d}{dt}(\mathbf{S}\mathbf{L}) = e(\mathbf{S} \times \mathbf{P}) \cdot (\mathbf{r} \times \mathbf{B}) \neq 0$ . The scalar  $\tilde{Q}'_2 = \mathbf{S}\mathbf{L}$  is the integral

of motion only if  $\mathbf{B} = f(r) \cdot \mathbf{r}$ ,  $r = \sqrt{\mathbf{r}^2}$ . This corresponds to the case of the monopole field, for which  $f(r) = gr^{-3}$  ( $\mathbf{r} \neq 0$ ,  $g = \text{const}$ ) is fixed by the condition  $\nabla \mathbf{B} = 0$ , and from now on, we restrict the analysis by the fermion-monopole system. Though in this case the brackets  $\{L_i, L_j\} = \epsilon_{ijk}(L_k + \alpha n_k)$ ,  $\alpha = eg$ , are different from the corresponding brackets for the free particle, nevertheless the odd scalar  $Q'_2 = \mathbf{L}\psi$  is the integral of motion. One can check that the direct analog of the free particle's supercharge (5) has zero bracket with  $Q_1$  and that

$$\{Q_2, Q_2\} = -i(\mathbf{J}^2 - \alpha^2). \quad (19)$$

Here  $\mathbf{J} = \mathbf{L} + \mathbf{S} - \alpha \mathbf{r} \cdot \mathbf{r}^{-1}$  is the conserved angular momentum vector of the fermion-monopole system, whose components form  $su(2)$  algebra,  $\{J_i, J_j\} = \epsilon_{ijk}J_k$ , and generate rotations. The scalar  $Q_3 = (\mathbf{P} \times \mathbf{L})\psi$  is also the integral of motion and in the fermion-monopole case the classical relations of the form (6), (7), (8) take place with the change  $\mathbf{J}^2 \rightarrow \mathbf{J}^2 - \alpha^2$  and with the Hamiltonian given by Eq. (18). Like for the free particle, the superalgebra can be reduced to the standard linear form (12) via the nonlinear transformation (11), for which in the present case we proceed from the relation  $\mathbf{J}^2 - \alpha^2 = 2H \cdot \Delta$  with  $\Delta = \tilde{\mathbf{r}}^2 + \mathbf{L}\mathbf{S} \cdot \mathbf{H}^{-1}$ ,  $\tilde{\mathbf{r}}^2 = \mathbf{r}^2 - (\mathbf{P}\mathbf{r})^2 \cdot (2H)^{-1}$ . The quantity  $\tilde{\mathbf{r}}^2$  is the integral of motion which in the case of the scalar charged particle ( $\psi_i = 0$ ) in the field of monopole gives a minimal charge-monopole distance in the point of perihelion:  $r_{\min} = \sqrt{\tilde{\mathbf{r}}^2}$  [8].

Constructing the quantum analogs of the supercharges in the same way as in the free particle case, we get the supercharge operators of the form (13) with  $\hat{\mathbf{p}}$  changed for  $\hat{\mathbf{P}}$ . They satisfy the set of (anti-)commutation relations of the same form (14)–(16) with  $\hat{A}_2 = \hat{\mathbf{J}}^2 + \frac{1}{4}$  changed for  $\hat{\mathbf{J}}^2 - \alpha^2 + \frac{1}{4}$ , where  $\alpha$  is subject to the Dirac quantization condition,  $2\alpha \in \mathbb{Z}$ .

The fermion-monopole integral  $\hat{Q}_2$  was observed for the first time on algebraic grounds by d'Hoker and Vinet [2] in the context of generalization of the so called dynamical symmetries of the charge-monopole system [10] for the supersymmetric case. The supercharge nature of  $\hat{Q}_2$  and the associated nonstandard nonlinear superalgebra of the operators  $\hat{Q}_2$  and  $\hat{Q}_1$ , was uncovered by De Jonghe, Macfarlane, Peeters and van Holten [1]. The present analysis shows that the set of supercharge operators  $\hat{Q}_1$  and  $\hat{Q}_2$  has to be extended by the scalar integral  $\hat{Q}_3$ , and these three odd operators together with even operators  $\hat{H}$  and  $\hat{\mathbf{J}}$  form the described nonlinear  $N = 3/2$  superalgebra. As we have seen, this nonlinear supersymmetry of the fermion-monopole system has the nature of the free fermion particle's supersymmetry generated by the supercharges represented in a scalar form.

Comparing the fermion-monopole Hamiltonian (18) (with  $\mathbf{B} = \alpha \mathbf{r} \cdot \mathbf{r}^{-3}$ ) with the free fermion particle Hamiltonian  $H = \frac{1}{2}\mathbf{p}^2$ , it seems that they have rather different structure, but this is not so, and their similarity can be revealed, like in the case of the scalar particle [8], by separating the even phase space coordinates into the radial and angular ones. The radial coordinates for the fermion-monopole system are  $r$  and  $P_r = \mathbf{P}\mathbf{n}$ , and the angular phase space variables are  $\mathbf{n} = \mathbf{r} \cdot \mathbf{r}^{-1}$ ,  $\mathcal{J} = \mathbf{L} - \alpha \mathbf{n}$ . These coordinates have the nontrivial brackets  $\{r, P_r\} = 1$ ,  $\{\mathcal{J}_i, \mathcal{J}_j\} = \epsilon_{ijk}\mathcal{J}_k$ ,  $\{\mathcal{J}_i, n_j\} = \epsilon_{ijk}n_k$ , and satisfy the relations  $\mathcal{J}\mathbf{n} = -\alpha$ ,  $\mathbf{n}^2 = 1$ . In terms of these variables, the Hamiltonian of the fermion-monopole system is

$$H = \frac{1}{2}P_r^2 + \frac{\mathbf{J}^2 - \alpha^2}{2r^2} - \frac{\mathbf{L}\mathbf{S}}{r^2},$$

with  $\mathbf{J} = \mathcal{J} + \mathbf{S}$ . The case of the free fermion corresponds to  $\alpha = 0$ , and its Hamiltonian

takes the similar form  $H = \frac{1}{2}p_r^2 + \frac{1}{2r^2}\mathbf{J}^2 - \frac{1}{r^2}\mathbf{LS}$ , where  $\mathbf{J} = \mathbf{L} + \mathbf{S}$ . The difference between the two systems is encoded now in the topology of even angular phase space variables [8].

**4.** Let us look at the fermion-monopole supersymmetry from the point of view of the reduction of the system to the spherical geometry. To this end we first note that the bracket of the supercharge  $Q_2$  with itself can be represented in the form

$$i\{Q_2, Q_2\} = 2r^2 \left( H - \frac{1}{2}P_r^2 + \frac{i}{r}Q_1(\psi\mathbf{n}) \right), \quad (20)$$

and the supercharge  $Q_3$  can be reduced to the equivalent form

$$Q_3 = 2Hr \left( \psi\mathbf{n} - Q_1 \frac{P_r}{2H} \right). \quad (21)$$

As it was shown in ref. [1], the reduction of the fermion-monopole system to the spherical geometry can be realized by introducing into the system the classical relations

$$\mathbf{r}^2 - \rho^2 = 0, \quad P_r = 0, \quad (22)$$

$$\psi\mathbf{n} = 0, \quad (23)$$

which have to be treated as the set of second class constraints with  $\rho \neq 0$  being a constant, and for simplicity we fix it in the form  $\rho = 1$ . The relations (20) and (21) allow us to observe directly that the described  $N = 3/2$  nonlinear fermion-monopole supersymmetry is transformed into the  $N = 1$  supersymmetry of the standard linear form in the case of reduction (22), (23). Indeed, after reducing the fermion-monopole system onto the surface of even second class constraints (22), we find that the structure of the supercharge  $Q_3$  is trivialized and takes the form of the odd scalar  $\psi\mathbf{n}$  multiplied by  $2H$ . Two other supercharges  $Q_1$  and  $Q_2$  after such a reduction take the form of linear combinations of the odd vector  $\psi$  projected on the vectors  $\mathcal{J} + \alpha\mathbf{n}$  and  $\mathcal{J} \times \mathbf{n}$  orthogonal to  $\mathbf{n}$ . Then taking into account the odd second class constraint (23) results in eliminating the supercharge  $Q_3$  and in reducing the bracket (and corresponding anticommutator at the quantum level) of the supercharge  $Q_2$  to  $2h$ , where  $h$  is the reduced Hamiltonian,

$$h = \frac{1}{2}(\mathbf{J}^2 - \alpha^2). \quad (24)$$

In other words, the supersymmetry of the fermion-monopole system in spherical geometry is reduced to the standard linear  $N = 1$  supersymmetry characterized by two supercharges anticommuting for the Hamiltonian [9]. More explicitly, after reduction to the surface of the second class constraints, the radial variables  $r$  and  $P_r$  are eliminated from the theory. The even variables can be represented by the total angular momentum  $\mathbf{J}$  and by the unit vector  $\mathbf{n}$  having the nontrivial Dirac brackets coinciding with corresponding initial Poisson brackets,  $\{J_i, J_j\}^* = \epsilon_{ijk}J_k$ ,  $\{J_i, n_j\}^* = \epsilon_{ijk}n_k$ . The odd variables  $\psi_i$  satisfy the relation (23) which has to be treated as a strong equality, and their nontrivial Dirac brackets are  $\{\psi_i, \psi_j\}^* = -i(\delta_{ij} - n_i n_j)$ ,  $\{J_i, \psi_j\}^* = \epsilon_{ijk}\psi_k$ . The even and odd variables are subject also to the relation  $\mathbf{J}\mathbf{n} = -\alpha + iq_1q_2 \cdot (2h)^{-1}$ , where  $h$  is given by Eq. (24) and

$$q_1 = (\mathbf{J} \times \mathbf{n})\psi, \quad q_2 = \mathbf{J}\psi \quad (25)$$

are the supercharges  $Q_1$  and  $Q_2$  reduced to the surface (22), (23). With the listed Dirac brackets, one can easily check that now the reduced supercharges (25) satisfy the superalgebra of the standard  $N = 1$  supersymmetry:  $\{q_\mu, q_\nu\}^* = -2i\delta_{\mu\nu}h$ ,  $\{q_\mu, h\} = 0$ ,  $\mu, \nu = 1, 2$ .

**5.** To conclude, let us compare the structure of supersymmetric quantum mechanics with the structure of the free fermion and fermion-monopole systems. A supersymmetric quantum mechanical system [9] is characterized by the Hamiltonian  $\hat{H} = \frac{1}{2}(\hat{p}^2 + W^2(x) + \sigma_3 W'(x))$  with  $\hat{p} = -id/dx$ , and by the supercharges  $\hat{Q}_1 = \hat{\theta}_1 W(x) - \hat{\theta}_2 \hat{p}$ ,  $\hat{Q}_2 = \hat{\theta}_2 W(x) + \hat{\theta}_1 \hat{p}$  with  $\hat{\theta}_\mu = \frac{1}{\sqrt{2}}\sigma_\mu$ ,  $\mu = 1, 2$ , which form the  $N = 1$  superalgebra  $[\hat{Q}_\mu, \hat{Q}_\nu]_+ = 2\delta_{\mu\nu}\hat{H}$ ,  $[\hat{H}, \hat{Q}_\mu] = 0$ . The operator  $\hat{Q}_3 = \frac{1}{\sqrt{2}}\sigma_3$  is the trivial integral of motion,  $[\hat{Q}_3, \hat{H}] = 0$ , and the set of supercharges  $\hat{Q}_{1,2}$  together with  $\hat{Q}_3$  satisfy the relations of the form (14) with  $\hat{A}_1 = \hat{A}_2 = 2\hat{B}_3 = 2\hat{H}$ ,  $\hat{A}_3 = \hat{B}_1 = \hat{B}_2 = 1$ . In this case the operator  $\sigma_3 = -2i\hat{\theta}_1\hat{\theta}_2 = \sqrt{2}\hat{Q}_3$  (being analogous to any component of the odd vector integral  $\sqrt{2}\hat{\psi}$  of the free fermion system) plays the role of the grading operator commuting (anticommuting) with operators  $\hat{x}$  and  $\hat{p}$  ( $\hat{\theta}_{1,2}$ ), and, as a consequence, commuting (anticommuting) with the Hamiltonian  $\hat{H}$  (supercharges  $\hat{Q}_{1,2}$ ). On the other hand, for the fermion-monopole and the free fermion systems, one can construct the operator  $\sqrt{2}\hat{\xi}_3$  proceeding from the classical relation (17). It seems that such operator could be treated as the grading operator due to the relation  $2\hat{\xi}_3^2 = 1$  and its anticommutation with the supercharges  $\hat{Q}_{1,2}$ . But such interpretation is not correct since unlike the case of supersymmetric quantum mechanics, this operator has a nontrivial dependence on even operators, and as a consequence, does not commute with them.

The ordinary form of the classical Lagrangian for the supersymmetric quantum mechanics,

$$L = \frac{1}{2}(\dot{x}^2 - W^2(x) - 2iW'(x)\theta_1\theta_2 + i\theta_\mu\dot{\theta}_\mu), \quad (26)$$

contains only two Grassmann variables. At the classical level the even quantity  $-2i\theta_1\theta_2$  corresponds to the odd operator  $\sigma_3$  (the latter being one of three generators of the corresponding Clifford algebra), i.e. here we have some sort of classical anomaly [11]. However, the symmetry between quantum and classical pictures can easily be restored (“the anomaly can be canceled”) extending the set  $\theta_\mu$ ,  $\mu = 1, 2$ , by the independent Grassmann variable  $\theta_3$  and changing Lagrangian (26) for  $\mathcal{L} = L + \frac{i}{2}\theta_3\dot{\theta}_3$ . Such a classical system has two nontrivial supercharges  $Q_1, Q_2$ , and the third odd integral of motion given by  $\theta_3$  (like  $\psi_i$  for the free fermion) is trivial and completely decoupled from other variables. Therefore, the difference of the superalgebraic structures of the supersymmetric quantum mechanics on the one hand and fermion-monopole system on the other hand is also reflected in the absence in the latter case of the grading operator commuting with the Hamiltonian and anticommuting with any two of three scalar supercharges and which simultaneously would commute with the initial coordinate and momenta operators.

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